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**CASE FILE
COPY**

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PART I

SUMMARY OF WORK ACCOMPLISHED DURING THE PERIOD

SEPTEMBER 1, 1971 - APRIL 30, 1972

During the period of September 1, 1971 through April 30, 1972, several improvements were made on the Compensator Improvement Program (CIP).¹ In the first interim period (September 1, 1971 - December 31, 1971) CIP was modified so that multi-channel systems and factored compensators could be handled. The factored compensators make allowances for the control of compensator pole-zero locations, and the factored compensators can be used to increase the convergence rate of the program. These two advantages of the factored compensators were demonstrated experimentally by using the CIP program to design a single channel compensator for a Saturn launch vehicle that had been previously designed using a single rational function form.²

In the second interim period (January 1, 1972 - April 30, 1972) finalization of the CIP was accomplished. In this time interval several additional options were included in the program. These options are the choice of allowing nonminimum phase compensators, the choice of perturbing some frequency points with respect to more than one point, and the choice of the program continuance mode, i. e., the total improved frequency response mode or the sum improved frequency response mode. The practical usefulness of CIP was demonstrated by designing compensators for the Saturn V/S1-C Dry Work Shop and the Saturn V/S1-C Sky Lab. From these examples it is pointed out that using CIP considerable improvement of compensators can be made, and the amount of effort to design compensators is reduced greatly.³

As a conclusion to CIP summaries of the subprograms were documented in May, 1972. This information was considered necessary in case some of the subprograms are to be modified or replaced by equivalent programs. These summaries present and explain the basic programming theory of the specialized subprograms (programs not normally found in a computing centersystem library). Also, complete descriptions of each I/O variable of each subprogram are given. With the programming theory and I/O variable information the modification or reproduction of any of these programs should be minimized.⁴

References

1. Jerrel R. Mitchell and Willie L. McDaniel, Jr., "Compensator Improvement with an Application to a Large Space Vehicle," Final Report, Contract No. NAS8-21377, NASA, MSFC, Huntsville, Ala., Covering Period September 1, 1969 - June 30, 1971.
2. Jerrel R. Mitchell and Willie L. McDaniel, Jr., "Extensions of Compensator Improvement to Include Multi-Channel Systems and Factored Compensators," Interim Report, Contract No. NAS8-21377, NASA, MSFC, Huntsville, Ala., Covering Period September 1, 1971 - December 31, 1971.
3. Jerrel R. Mitchell and Willie L. McDaniel, Jr., "An Innovative Approach to Compensator Design," Interim Report, Contract No. NAS8-21377, NASA, MSFC, Huntsville, Ala., Covering Period January 1, 1972 - April 30, 1972.
4. Jerrel R. Mitchell and Willie L. McDaniel, Jr., "Subprogram Summaries of Compensator Improvement Program," Monthly Report, Contract No. NAS8-21377, NASA, MSFC, Huntsville, Ala., Covering Period May 1, 1972 - May 31, 1972.

PART II

MAGNITUDE ERROR BOUNDS FOR SAMPLED-DATA FREQUENCY RESPONSE
OBTAINED FROM THE TRUNCATION OF AN INFINITE SERIES

Introduction

In order to properly design a control system it is first necessary to be able to analyze the performance. In general there are three basic classical approaches for analyzing control systems; these approaches are the frequency response method, the time domain method, and the root locus method. These methods all transmit the same information. However, deciphering some information which is obvious while using one technique can be very difficult while using either of the other techniques. Thus, each method has certain advantages and disadvantages which usually depend upon the type of system performance to be measured.

Probably the most widely used technique of analyzing control system performance is by the frequency response method. The reasons that this technique is so popular are the simplicity of obtaining frequency response information with the aid of the digital computer and the availability of experimental frequency response information.

The frequency response method has been of great utility in the analysis of sampled-data control systems that are unity feedback and error sampled. However, one major problem that is associated with the application of the frequency response method to sampled-data control system analysis is the difficulty of obtaining the sampled frequency response of the continuous system. A technique for accomplishing this is given by the following infinite series:

$$G^*(j\omega) = \frac{1}{T} \sum_{q=-\infty}^{\infty} G(j\omega + jq\omega_s) + g(0)/2 \quad (1)$$

where $G^*(j\omega)$ is the sampled frequency response, $G(j\omega)$ is the frequency response of the continuous system, T is the sampling period, ω_s is the sampling frequency and ω is the radian frequency.¹ Practical use of (1) is made by truncating the series after a certain number positive and negative terms, e.g., q 's on the interval of $-k \leq q \leq k$ where k is a positive integer. The result of performing this produces the following finite series:

$$\bar{G}^*(j\omega) = \frac{1}{T} \sum_{q=-k}^k G(j\omega + jq\omega_s) + g(0)/2 \quad (2)$$

in which $\bar{G}^*(j\omega)$ is the approximate sampled frequency response. It then follows that

$$\bar{G}^*(j\omega) \approx G^*(j\omega) \quad (3)$$

Upper Bound Magnitude Error Derivation

It is important to know how many positive and negative terms of the series must be taken so that $\bar{G}^*(j\omega)$ is a reasonable approximation to $G^*(j\omega)$ for $-\frac{\omega_s}{2} < \omega < \frac{\omega_s}{2}$.^{*} One technique for studying this is to determine the magnitude error between the two frequency responses. This error is defined as

$$|E(j\omega)| = |G^*(j\omega) - \bar{G}^*(j\omega)| \quad (4)$$

where $E(j\omega)$ is the error between the actual sampled frequency response and the approximate sampled frequency response. Subtracting (2) from (1),

^{*}It is only necessary to consider the frequency range $-\frac{\omega_s}{2} \leq \omega \leq \frac{\omega_s}{2}$, which defines the primary strip in the $G^*(s)$ -plane, since the frequency response $G^*(j\omega)$ is simply duplicated for frequencies in each secondary strip of the $G^*(s)$ -plane.

(4) becomes

$$|E(j\omega)| = \frac{1}{T} \left| \sum_{q=k+1}^{\infty} G(j\omega + jq\omega_s) + \sum_{q=-k-1}^{-\infty} G(j\omega + jq\omega_s) \right| \quad (5a)$$

or

$$|E(j\omega)| = \frac{1}{T} \left| \sum_{q=k+1}^{\infty} [G(j\omega + jq\omega_s) + G(j\omega - jq\omega_s)] \right| \quad (5-b)$$

In order to determine the magnitude error exactly it is necessary to evaluate the infinite series given by (5a) or (5b). This is impractical and actually defeats the purpose for truncating the original series. However, if an easy approach to calculating a reasonable upper bound on the magnitude error can be established, then a useful relation exists.

Consider the continuous frequency response $G(j\omega)$ and suppose that for all $\omega \geq k\omega_s$ and for all $\omega \leq -k\omega_s$ that

$$|G(j\omega)| \leq M/|\omega|^n \quad (6)$$

where M is an appropriately chosen constant and n is the difference in the number of poles and zeros of $G(s)$.^{*} This assumption is certainly not unreasonable if $k\omega_s$ is greater than all break frequencies of $G(j\omega)$. Next assume that $n \geq 1$.

Using the identity for complex numbers, z_1, z_2, \dots, z_j which states that $|z_1 + z_2 + \dots + z_j| \leq |z_1| + |z_2| + \dots + |z_j|$ (5) becomes

^{*} It should be obvious that (6) assumes $G(j\omega) \approx M/(j\omega)^n$ for all $\omega \geq k\omega_s$ and for all $\omega \leq -k\omega_s$.

$$\begin{aligned}
|E(j\omega)| &= \frac{1}{T} \left| \sum_{q=k+1}^{\infty} [G(j\omega + jq\omega_s) + G(j\omega - jq\omega_s)] \right| \\
&\leq \frac{1}{T} \sum_{q=k+1}^{\infty} |G(j\omega + jq\omega_s) + G(j\omega - jq\omega_s)| \\
&\leq \frac{1}{T} \sum_{q=k+1}^{\infty} [|G(j\omega + jq\omega_s)| + |G(j\omega - jq\omega_s)|] \quad (7)
\end{aligned}$$

Using (6), the magnitude error is bounded by

$$|E(j\omega)| \leq \frac{M}{T} \sum_{q=k+1}^{\infty} \left[\frac{1}{(q\omega_s + \omega)^n} + \frac{1}{(q\omega_s - \omega)^n} \right] \quad (8)$$

in which it has been assumed that $q\omega_s > \omega$ since $\omega \in [-\frac{\omega_s}{2}, \frac{\omega_s}{2}]$.

The result is that the magnitude error can be bounded by the sum of two infinite series. It is possible to bound either of these series from above and below by Riemann Zeta series minus certain constants.* Since either of the series in (8) can be bound from below by a Riemann Zeta series then neither series converges for $n=1$ because for $n=1$ Riemann Zeta series diverge. Thus the above bound is only applicable for system's with at least two more poles than zeros, i. e., $n \geq 2$.

Returning to (8) it is observed that both series are positive series. Thus the values of each series is less than the integral between the

*A Riemann Zeta series or p-series is an infinite series of the form $\sum_{k=1}^{\infty} 1/k^n$. A series of the form $\sum_{q=k+1}^{\infty} \frac{1}{(q\omega_s + \omega)^n}$ where $\omega = p\omega_s$ and $p \in [p, 0.5]$ is bound by $\frac{1}{\omega_s^n} \sum_{q=k+1}^{\infty} \frac{1}{q^n} \geq \sum_{q=k+1}^{\infty} \frac{1}{(q\omega_s + \omega)^n} \geq \frac{1}{\omega_s^n} \sum_{q=k+1}^{\infty} \frac{1}{(q+1)^n}$.

The series on the extremes of this inequality are simply weighted Riemann Zeta Series with the first $(k-1)$ and k terms, respectively, subtracted off. The subtracted terms form the constants mentioned above. The series

$\sum_{q=k+1}^{\infty} \frac{1}{(q\omega_s - \omega)^n}$ may be bound in a similar manner.

limits of k and ∞ of the continuous curves fabricated by replacing each summing index by a continuous variable.² In mathematical notation this produces

$$\begin{aligned}
 |E(j\omega)| &\leq \frac{M}{T} \sum_{q=k+1}^{\infty} \left[\frac{1}{(q\omega_s + \omega)^n} + \frac{1}{(q\omega_s - \omega)^n} \right] \\
 &\leq \frac{M}{T} \left[\int_k^{\infty} \frac{dx}{(x\omega_s + \omega)^n} + \int_k^{\infty} \frac{dx}{(x\omega_s - \omega)^n} \right] = \\
 &\quad \frac{M}{T} \left(\frac{1}{\omega_s(-n+1)} \left[\frac{1}{(x\omega_s + \omega)^{n-1}} + \frac{1}{(x\omega_s - \omega)^{n-1}} \right] \right) \Bigg|_k^{\infty} \quad (9a)
 \end{aligned}$$

or

$$|E(j\omega)| \leq \frac{M}{2\pi(n-1)} \left[\frac{1}{(k\omega_s + \omega)^{n-1}} + \frac{1}{(k\omega_s - \omega)^{n-1}} \right] \quad (9b)$$

The expression given in (9) can be used to calculate an upper bound magnitude error for any $\omega \in [-\frac{\omega_s}{2}, \frac{\omega_s}{2}]$. It is obvious that $|E(j\omega)|$ is an even function with respect to ω , i.e., $|E(j\omega)| = |E(-j\omega)|$. It is then only necessary to consider ω 's on the range $0 \leq \omega \leq \frac{\omega_s}{2}$.

For ω 's on this range it is easily deduced that the maximum value of the upper bound magnitude error occurs at $\omega = \omega_s/2$. Thus the absolute maximum magnitude error that any point on this given frequency range can achieve is

$$E_{\max} = \frac{M}{2\pi\omega_s^{n-1}(n-1)} \left[\frac{1}{(k + 1/2)^{n-1}} + \frac{1}{(k - 1/2)^{n-1}} \right] \quad (10)$$

The relation (10) allows for the calculation of a bound on the maximum magnitude error that any point in the allowable frequency range can possess. A quick estimate of the magnitude error of any frequency point,

where $\omega \in [0, \omega_s/2]$, can be obtained from (10), whereas (9) can be used to obtain a point by point maximum magnitude error for frequency points on the interval $0 \leq \omega \leq \omega_s/2$.

It is obvious from both (9) and (10) that the choice of M affects the maximum magnitude error upper bound. In fact the smaller M is, the smaller the upper bound error. Thus, the closer that $M/|\omega^n|$ approximates the magnitude of $G(j\omega)$ for $\omega \geq k\omega_s$ and $\omega \leq -k\omega_s$ the more accurate the upper bound magnitude error will be in predicting the actual error. One technique for calculating the value of the smallest M is to plot the $|G(j\omega)|$ for $\omega > k\omega_s$ on log - log paper and then construct a straight line whose slope is $[-(\text{octave}) \times n/\text{octave}]$ which is greater than $|G(j\omega)|$ for $\omega > k\omega_s$ but is as near as possible to $|G(j\omega)|$. This straight line may be labeled as $S(\omega)$. From the log - log plot for some value of frequency, e.g., $\omega = \omega_1$, the corresponding value of $S(\omega)$ can be read, i.e., $S(\omega_1)$. Then $M = S(\omega_1) \cdot \omega_1^n$.

Examples

The previously derived upper bound magnitude error for bounding the magnitude error caused by truncating the infinite series, (1), after k positive and negative terms is of practical use only if the bound can produce a reasonable estimate of the actual error. In order to illustrate that the upper bound calculation given in (9b) can do this, two examples of its use are given in the following:

Example 1 - Consider a continuous system described by the following transfer function:

$$G(s) = \frac{1.0}{(s+1)^2} .$$

It is desired to use (9b) to approximate the sampled frequency response of this system when $\omega_s = 25$ rad/sec and $k = 2$. Table 1 is used to compare the approximate frequency response to the exact frequency response and to compare the actual magnitude error to the upper bound magnitude error calculated from (9b) when $M = 1.0$.*

Example 2 - In this case consider a continuous system described by

$$G(s) = 256.0 \frac{(s+2)}{(s+4)(s+8)(s+16)}$$

For $k = 5$, $\omega_s = 25$ rad/sec, and $M = 256.0$ Table 2 can be used to make comparisons between the approximate and the exact frequency responses and the actual and the upper bound magnitude errors.

Analysis of Results and Conclusion

From the preceding examples it is seen that the upper bound magnitude error equation, (9b) can yield reasonable estimates of the actual error. However, it should be pointed out that upper bound calculations depend very much upon M . In fact in the examples given the upper bound calculations are as small as possible because M was chosen so that $M/|\omega^n|$ is the actual asymptote of $|G(j\omega)|$ as $\omega \rightarrow \infty$. If $k\omega_s$ had been small enough so that this would not have been possible then the upper bound calculation would have probably been much greater than the actual error, e.g., cases where $k\omega_s$ is less than the last break frequency of $G(j\omega)$ or the value of $k\omega_s$ is close to the break frequency of a pair of complex conjugate poles with $\delta < 0.707$.

* Magnitude error is defined in (4).

There are two extensions or modifications which would greatly improve upper bound magnitude calculations. First, the technique derived in this report is not applicable to systems with only one more pole than zero. Thus the upper bound magnitude derivation should be extended so that this case is included. Secondly, modifications of the derivation should be made so that a tighter bound can be calculated. This could be done by producing tables that could be used for the actual evaluation of (8). This would eliminate the error caused by approximating the value of the infinite series of (8) by integrals of continuous variables. With these extensions the upper bound magnitude calculation would be enhanced greatly.

References

1. Benjamin C. Kuo, Analysis and Synthesis of Sampled-Data Control Systems, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1963, pp. 34-35.
2. George B. Thomas, Calculus and Analytic Geometry--Part 2, Addison-Wesley Publishing Company, Reading, Massachusetts, 1964, pp. 777-780.

FREQUENCY IN RAD/SEC	APPROX. MAGNITUDE	APPROX. ANGLE IN DEG.	EXACT MAGNITUDE	EXACT ANGLE IN DEG.	MAGNITUDE ERROR	UPPER BOUND MAGNITUDE ERROR
0.0	3.96302	0.00000	3.95812	0.00000	0.00490	0.00637
1.0	1.98937	- 90.46503	1.98941	- 90.60621	0.00490	0.00637
2.0	0.80533	-127.80748	0.80835	-128.08244	0.00491	0.00638
3.0	0.41092	-144.56148	0.36183	-147.77504	0.00491	0.00639
4.0	0.24886	-153.88602	0.25328	-154.37811	0.00492	0.00641
5.0	0.16912	-159.91012	0.17375	-160.47279	0.00493	0.00643
6.0	0.12481	-164.23144	0.12957	-164.83240	0.00494	0.00646
7.0	0.09811	-167.58600	0.10295	-168.18952	0.00496	0.00649
8.0	0.08118	-170.35874	0.08609	-170.92757	0.00498	0.00653
9.0	0.07020	-172.77083	0.07516	-173.26851	0.00500	0.00658
10.0	0.06314	-174.95901	0.06814	-175.35246	0.00502	0.00663
11.0	0.05889	-177.01348	0.06394	-177.27558	0.00505	0.00669
12.5	0.05663	-179.98029	0.06172	-180.01326	0.00510	0.00679

Table 1. Frequency Response Information for
Example 1

FREQUENCY IN RAD/SEC	APPROX. MAGNITUDE	APPROX. ANGLE IN DEG.	EXACT MAGNITUDE	EXACT ANGLE IN DEG.	MAGNITUDE ERROR	UPPER BOUND MAGNITUDE ERROR
0.0	1.16956	0.00000	0.61611	0.00000	0.55345	0.65190
1.0	1.50596	13.36757	0.97617	20.94539	0.55350	0.65196
2.0	2.05021	5.47104	1.50030	7.54703	0.55357	0.65207
3.0	2.42480	- 10.20970	1.88187	- 13.12827	0.55372	0.65227
4.0	2.58375	- 27.43201	2.10583	- 34.31560	0.55396	0.65257
5.0	2.57781	- 44.52567	2.21379	- 54.56196	0.55420	0.65294
6.0	2.46715	- 61.22314	2.24866	- 73.64640	0.55453	0.65340
7.0	2.29979	- 77.71393	2.24248	- 91.67512	0.55492	0.65395
8.0	2.11052	- 94.35320	2.21668	-108.83301	0.55537	0.65458
9.0	1.92478	-111.55116	2.18492	-125.31041	0.55588	0.65530
10.0	1.76251	-129.68568	2.15561	-141.28807	0.55645	0.65610
11.0	1.64062	-148.96898	2.13380	-156.90948	0.55708	0.65699
12.5	1.56277	-179.64632	2.12080	-180.01321	0.55815	0.65848

Table 2. Frequency Response Information for Example 2